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The Model Method in Singapore

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Abstract: *Since its introduction in the primary schools in Singapore more than fifteen years ago by Dr Kho Tek Hong and his team, the power of the "model method" has been tapped to solve many challenging arithmetic word problems, and also to enable primary school pupils to solve questions that were previously set for secondary school students. As learning and teaching the model method continues to be encouraged in many schools, we look at some common difficulties associated with the model method, and suggest how we may help pupils overcome them.*

Introduction

Although the model method, introduced in 1983, is well accepted today by most primary school teachers, many average pupils and educated parents still find drawing block models difficult, especially when it comes to solving the word problems in arithmetic (Kho, 1987). In this article, some difficulties commonly encountered in drawing models, in particular, the following, are discussed:

- Difficulty of an "accurate diagram";
- Divisions in a block diagram;
- Inappropriate use of the model method.

The model method gained popularity in Singapore, because it empowers pupils to solve mathematics problems that were traditionally set only at higher levels (Fong, 1994, 1999a, 1999b; Ng & Lim, 2001). The routine problems that many parents used to solve during their secondary school days, have now become the non-routine problems that their children and grandchildren are solving in their primary school years.

The difficulty of an accurate diagram

In drawing models for arithmetic word problems, it is not uncommon that the difficulty lies in drawing an "accurate" diagram. An accurate diagram is used here in the sense that the block diagram need not be scaled, but it should be good enough to enable one to deduce meaningful relationships between the known and unknown quantities. It can be argued that once you are able to draw a model, you have already understood the problem.

The basic unit of the model, whose value is to be found, may turn out to be shorter or longer than a known part in the model. Merely reading the model solution given in assessment books conceals the messy work usually involved in drawing an accurate model.

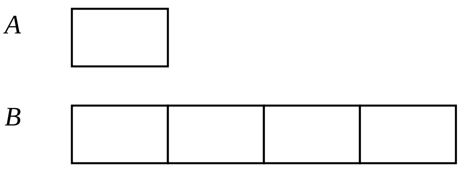
Let us look at some examples to illustrate why this crucial part of problem solving – drawing an “accurate diagram” – is important, if we are to gain a better insight of the mathematical concepts involved.

Example 1

A is one-quarter as old as *B*. In 5 years' time, *A* will be one-third as old as *B*. How old is *A* now?

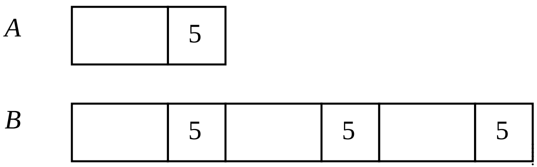
Let us approach the above problem as follows:

Present

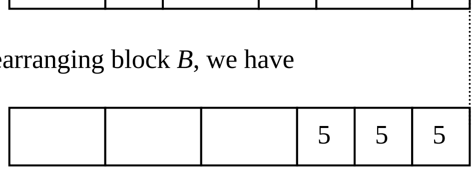


(1)

In 5 years' time

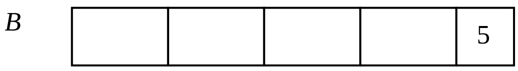


On rearranging block *B*, we have



(2)

Also, from (1), block *B* in 5 years’ time looks as follows:



(3)

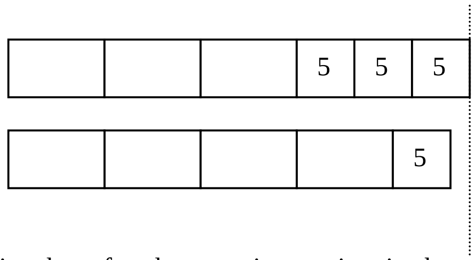
By comparing block *B* in (2) and (3), we have

$$\boxed{} = \boxed{5} + \boxed{5} = 10$$

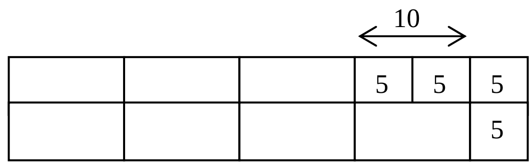
So, *A*'s present age is 10 years old.
Note that the block diagrams in (2) and (3), each representing *B*, turn out to be of different lengths, although they are expected to be equal.

A (2)

B (3)



This is what often happens in practice, in the process of drawing and comparing block diagrams. Yet, most, if not, all assessment books will depict the two blocks to be of equal length, as shown below:



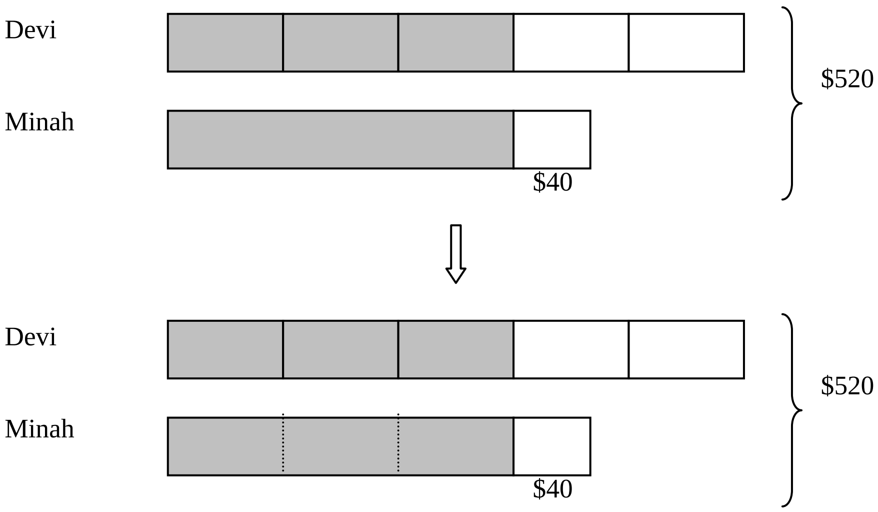
This means that the initial length of one unit used for blocks *A* and *B* were too short, as compared to the part representing 5 years. We need to lengthen the unit so that blocks *A* and *B* can be of equal length. We have to point out to pupils that assessment books hardly ever highlight this difficulty in modelling their solution. Because writers usually know the answer to the question posed, so they often work backwards to present a refined version of the models, skipping the crucial stages of thinking processes necessary to draw them.

Let us look at another example, which shows that representing the length of a block unit correctly, relative to any known part in the model, is not always easy.

Example 2

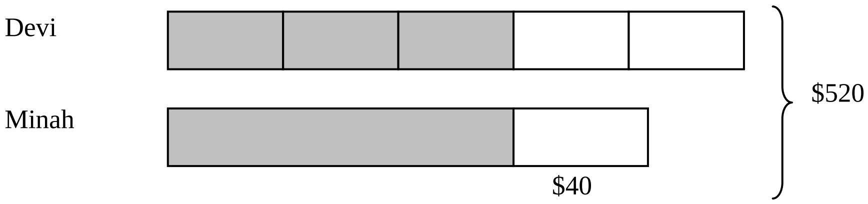
Devi and Minah have \$520 altogether. If Devi spends $\frac{2}{5}$ of her money and Minah spends \$40, then they will have the same amount of money left. How much money does Devi have? (Kho, 1987)

Solution:



8 units = \$520 - \$40 = \$480
1 unit = \$480 ÷ 8 = \$60
Devi's money = 5 units
= 5 x \$60
= \$300

However, if we have assumed that one unit is less than \$40, then our model will look as follows:



On comparing and simplifying, we have

1 unit = \$60
Devi's money = 5 units
= 5 x \$60
= \$300

It is clear that one unit in our second model, computed to be \$60, should look longer than the part in the block representing \$40.

So, it does not seem to matter whether the unit is taken initially to be longer or shorter than a known part in the block diagram, because in both cases, the answer is not affected. However, it does matter if we want the block diagram to look realistic – the partitions representing the units and the known part(s) must be sensibly proportionate. A scaled block diagram is not required, but the model must be sensibly drawn to a certain degree of proportionality so that any relationship between the units and any known part(s) may be deduced.

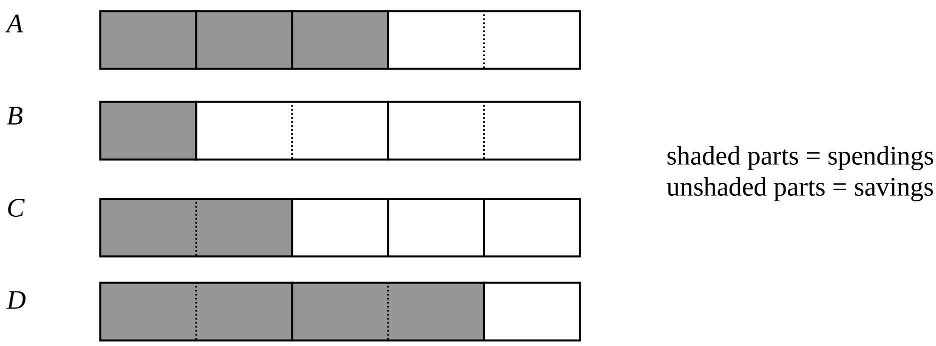
Let us turn to one more problem where a fair bit of trial-and-error is necessary, before the correct model may be drawn.

Example 3

> *A*, *B*, *C* and *D* earned the same amount of money every month. *A* spent three times as much as *B*. *D* spent twice as much as *C*. *B* saved twice as much as *A*. *C* saved three times as much as *D*. Find the ratio of *B*'s savings to *D*'s savings.

A typical solution to the above question, given by most assessment books, is as follows:

Solution



1 unshaded part = 2 shaded parts
= 2 units

So, each block has 5 units.
Ratio of *B*'s savings to *D*'s savings = 4 : 1

The above solution is generally not useful, even to experienced problem solvers, as the whole process of problem solving is not reflected. The presentation of solution skips the first stage of Polya's problem solving heuristics – understanding.

We often assume that pupils understand the problem immediately after reading it. The solution, which represents the last iterative cycle of the Polya's heuristic, hides all the reflection - thinking processes - that goes on, where much trial-and-error, or refining, is involved, before a fairly accurate diagram may be drawn. Teachers and writers often fail to discuss the workings involved in presenting a particular model.
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Thinking Processes

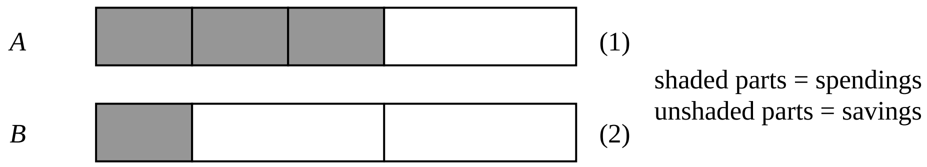
Let us see how the given solution to Example 3 could have been presented in a way that demonstrates some of the thinking processes needed to arrive at the answer.

The difficulty lies in drawing an "accurate model". First, we draw a sketch to study the relationship between the different parts, before drawing an "accurate diagram".

Given that

A spent three times as much as *B*, and
B saved twice as much as *A*,

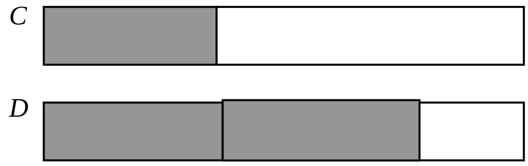
we construct a model diagram, using the relationships between *A* and *B*.



Similarly, given that

D spent twice as much as *C*, and
C saved three times as much as *D*,

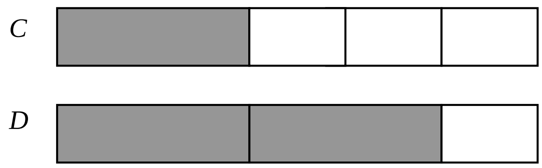
we construct a model diagram, using the relationships between *C* and *D*.



shaded parts = spendings
unshaded parts = savings

Since *A*, *B*, *C* and *D* each earned the same amount of money, the block diagrams for *C* and *D* must be of the same length as those of *A* and *B*. Note that although *D* spent twice as much as *C*, as depicted by the shaded parts in the block diagrams, the unshaded parts fail to satisfy the condition that *C* saved three times as much as *D* – the unshaded part of *C* is not three times the unshaded part of *D*.

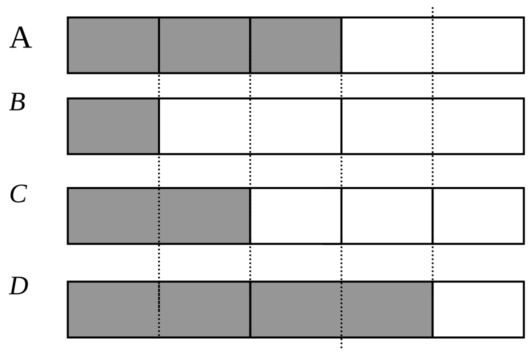
So, we need to adjust both the shaded and the unshaded parts in both blocks, until the two conditions governing *C* and *D* are satisfied, resulting in the shaded part being now longer than the shaded part in the previous case, as shown overleaf:



(3)

(4)

On comparing block diagrams *A*, *B*, *C* and *D*, labelled by (1), (2), (3), and (4), and using dotted lines, it is not difficult to see that each bar could be partitioned into five units for meaningful comparisons.



shaded parts = spendings
unshaded parts = savings

From the diagram,
Ratio of *B*'s savings to *D*'s savings = 4 : 1

So, it is clear that merely showing a modelled solution is of little use, especially for pupils and parents who are learning to apply the model method to solve word problems, unless we can model some of the processes leading to the final product.

The divisions in a block diagram

The numerical values found in many word problems that lend themselves to models, are relatively small, to allow pupils to handle them with ease. By dividing the block diagrams into concrete parts, sometimes up to 20 or 24 divisions, pupils are able to compare the units with the known part(s) quite easily, to arrive at the correct answer.

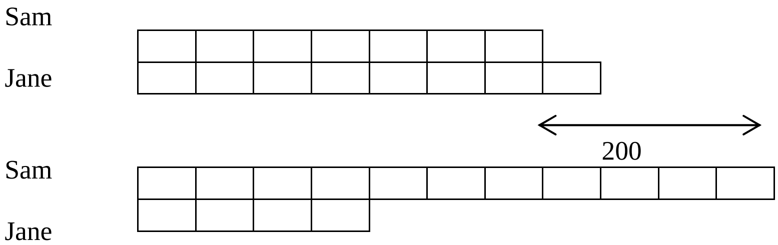
Let us look at one such question where most pupils can solve it easily, by dividing the block diagrams into an equal number of divisions, based on the numbers given in the problem.

Example 4

Sam and Jane have stamps in the ratio of 7 : 8. Jane gives Sam 200 of his stamps, so that the new ratio of Sam to Jane is now 11 : 4. How many stamps are there together?

Solution:

Before



For Sam:

Difference in units = 11 - 7
= 4

4 units = 200 stamps

Total number of units = 7 + 8
= 15

15 units = $(\frac{200}{4} \times 15)$ stamps
= 750 stamps

There are altogether 750 stamps.

While we should encourage pupils to use concrete divisions of a block diagram to deduce any relationship, which often leads instantly to the answer, we should also help them to slowly move away from having to rely on a large number of divisions, as these make comparisons difficult.

An overemphasis on dividing the block diagrams into a large number of parts to arrive at an answer, often short-circuits the thinking processes that would otherwise have been needed to abstract deeper relationships between the various parts of a model. One way to help pupils to depend less on drawing too many concrete divisions, is to use relatively large figures in the questions we set, by making it cumbersome to divide the block diagram into many parts.

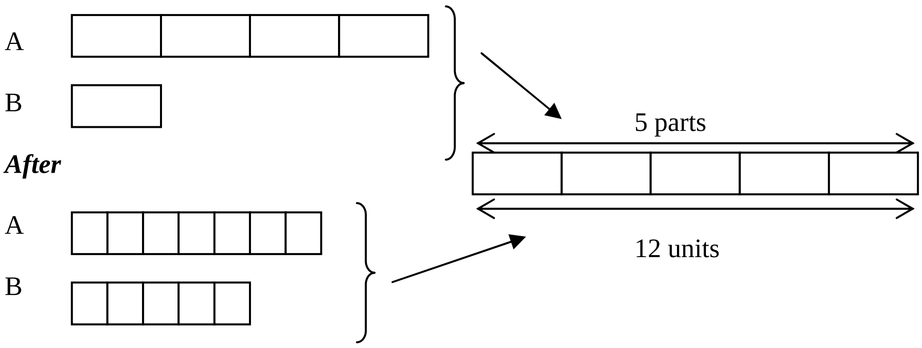
Example 5 typifies the difficulty and the impracticality of partitioning block diagrams into concrete parts to compare the units and the known parts in a model.

Example 5

There are two bags of sweets, A and B. The ratio of the mass of A to the mass of B is 4 : 1. If 10 g of sweets is transferred from A to B, the ratio of the mass of A to the mass of B is 7 : 5. Find the total mass of the two bags of sweets.

Solution:

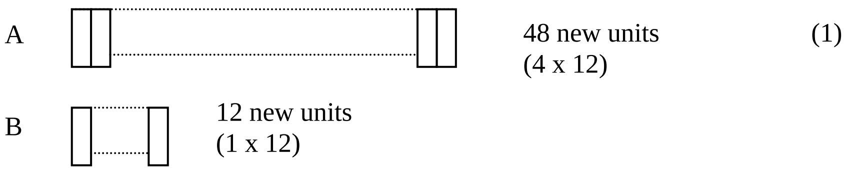
Before



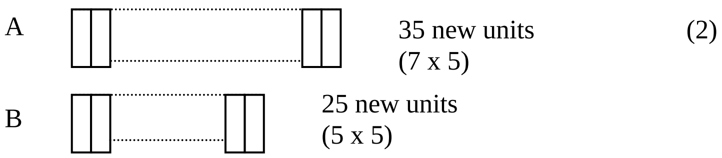
At this point, if we want to divide both blocks into equivalent blocks of comparable units, we need to divide the combined length of blocks A and B into exactly (5 x 12) units, or 60 units.

To avoid dividing the blocks into this large number of parts, we use the following equivalent block diagrams, by using dotted lines to symbolise the many divisions in between the first unit and the last unit.

Equivalent (Before) Block Diagram:



Equivalent (After) Block Diagram:



Comparing block A shown in (1) and (2), we have

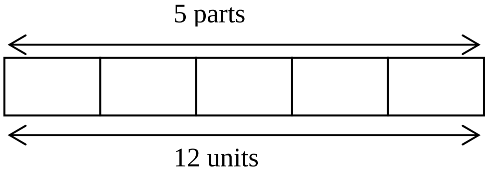
(48 - 35) new units = 10 g
13 new units = 10 g

(48 + 12) new units = 60 units $= \left(\frac{10}{13} \times 60\right)$ g

$= 46\frac{2}{13}$ g

The total mass of the two bags of sweets is $46\frac{2}{13}$ g.

Alternatively, we may obtain the answer as follows:



12 units = 5 parts

7 units = $(\frac{5}{12} \times 7)$ parts = $\frac{35}{12}$ parts

Comparing block A *before* and *after*, we have

$(4 - \frac{35}{12})$ parts = 10 g

$\frac{13}{12}$ parts = 10 g

5 parts = $(10 \times \frac{12}{13} \times 5)$ g

= $46\frac{2}{13}$ g

Total mass of the two bags of sweets = $46\frac{2}{13}$ g

The inappropriate use of the model method

The urge to educate and impress the lay public of the power and beauty of the model method, has led some writers to jump into the “model method” wagon, by churning out assessment books that often oversell the model approach (Fong, 1993; Ho, Ho, & Ong, 2001; Leong, 1990, 1991a, 1991b, 1991c; Markandoo & Mah, 2002; Ng, 1995a, 1995b, 1995c).

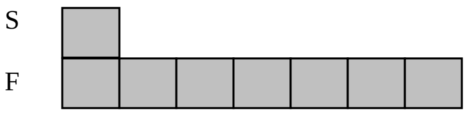
Consider the following age problem, where the method of solution apparently uses a model approach, but actually uses an algebraic approach instead.

Example 6

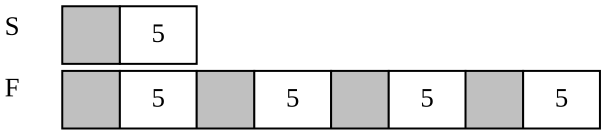
Two years ago, a man was 7 times as old as his son, but in three years’ time, he will be 4 times as old as the boy. How old is each of them now?

Solution 1

2 years ago S ≡ Son’s age, F ≡ Father’s age



3 years' time



Father's age

$7\square + 5 = 4\square + 20$ (1)

$7\square - 4\square = 20 - 5$

$3\square = 15$

$\square = 5$

Present

Son's age = $\square + 2$

= (5 + 2) years old
= 7 years old

Father's age = $7\square + 2$

= (7 x 5 + 2) years old
= 37 years old

Consider the block statement in (1): $7\square + 5 = 4\square + 20$

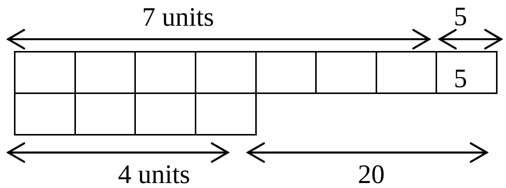
If we replace $\square$ by *x*, we have the algebraic equation: 7*x* + 5 = 4*x* + 20.

Since pupils are not required to solve such an algebraic equation in primary schools (Ministry of Education, 2001), the algebraic method used to find the value of the basic unit is inappropriate – be it in the explicit or hidden form of a model.

However, such algebraic approach to finding the value of the unit is perfectly appropriate in secondary schools.

Let us look at the block statement in (1) again: $7\boxed{} + 5 = 4\boxed{} + 20$

If we draw side by side two blocks, we have the following:



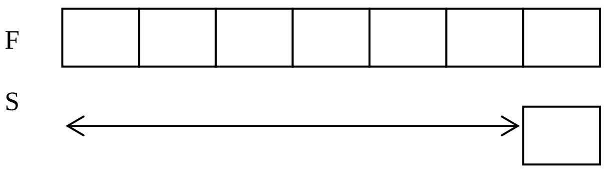
From the diagram, 3 units = 15
1 unit = 5

Would it be fair to expect pupils to be able to interpret the block statement, by drawing two block diagrams to find the value of the unit?

Let us look at another method of solution that begins with block diagrams, but ends up, using an algebraic approach to find the value of the basic unit.

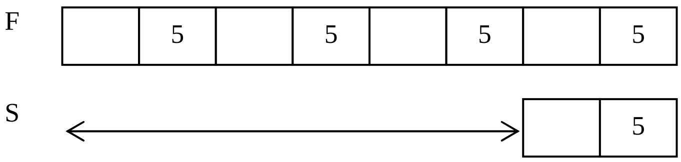
Solution 2

2 years ago F ≡ Father's age, S ≡ Son's age



Difference = 6

***5 years from then* (*i.e. in 3 years' time*)**



Difference = 3 $\boxed{}$ + 15

The difference in age remains the same.

$$6\,\boxed{} = 3\,\boxed{} + 15$$

$$3\,\boxed{} = 15$$

$$\boxed{} = 5$$

Present

Son's age = $\boxed{}$ + 2 = 7

Father's age = 7 $\boxed{}$ + 2 = (7 x 5) + 2 = 37

Clearly, if we replace each unit by *y* in the following statement:

$$6\,\boxed{} = 3\,\boxed{} + 15$$

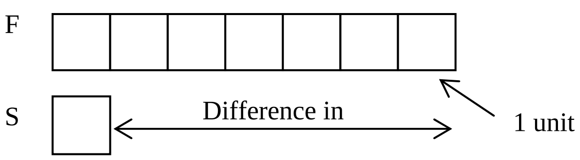
we end up with the algebraic equation, 6*y* = 3*y* + 15.

Both solutions 1 and 2 appear to use explicitly a model approach, but implicitly rely on an algebraic approach to arrive at the answer.

Let us see how a non-algebraic approach may be possible to solve the age problem.

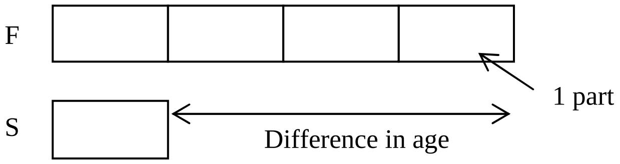
Solution:

2 years ago F ≡ Father, S ≡ Son



(1)

***5 years from then* (*i.e. in 3 years' time*)**



(2)

Difference in age (3 years later) = (4 – 1) parts
= 3 parts

Difference in age (2 years ago) = 6 units

So, 3 parts = 6 units
1 part = 2 units

Now, comparing the son's age from the two blocks shown in (1) and (2), we have

$$\boxed{} = \boxed{} + 5$$

1 part = 1 unit + 5

Since 1 part = 2 units, we have:

2 units = 1 unit + 5
1 unit = 5

Present

Father's age = 7 units + 2

= (7 x 5 + 2) years old

= 37 years old

Son's age = 1 unit + 2

= (5 + 2) years old

= 7 years old

Alternatively,

Father's age = 4 parts – 3

= (4 x 10 – 3) years old

= 37 years old

Son's age = 1 part – 3

= (10 – 3) years old

= 7 years old

Conclusion and Recommendations

The model method remains a powerful problem-solving tool to solve many challenging arithmetic word problems. Mathematical models help pupils gain concrete experiences which are pre-requisites for understanding abstract symbols of mathematics and their manipulation (Kho, 1982). Besides, the model method provides many opportunities to use heuristics such as “Draw a diagram”, “Use a model”, or ‘Use visualisation”.

However, model solutions presented in assessment books often conceal the difficulty in drawing an accurate model. As a result, weak pupils find it difficult to master the skills of drawing mathematical models, thus depriving them of concrete experiences to grasp mathematical concepts. To help pupils appreciate the thinking processes involved in drawing a fairly accurate model, assessment writers and teachers need to show that drawing a model is often a complex activity that involves quite a bit of redrawing and refining, before a useful model may be drawn.

Moreover, in drawing a fairly “accurate diagram”, the model needs to look realistic, with the units and known parts to be sensibly proportionate to enable any meaningful relationship to be deduced. We should also make pupils aware that a mathematical model - an “accurate diagram” - provides a powerful visual aid for them to better appreciate with the problem at hand.

Many parents who did not learn the model method during their school days, find themselves unable to coach their children; many of them turn to tutors for help. But most tutors themselves rely on assessment books to coach their pupils; many do not have a systematic approach to effectively use the model method to solve the harder word problems. It is not uncommon - and surprising - when we hear of parents complaining of tutors who are mathematics majors, being unable to solve many of the word problems in assessment books.

Teachers and writers have promoted the model approach, by mostly offering only one method of solution when, more often than not, several model approaches exist to solve the same problem. Such restrictive approach to problem solving gives pupils the impression that the model solution offered in the assessment book is the *only* correct one - that there are no other equally valid solutions that tap the power of the model method. It is common to hear pupils, or parents, complain that they have got the correct answer, but their model looks different from the one modeled in the book, which many find difficult to understand. Teachers and tutors could help pupils to come up with different models, if possible, to solve a particular problem. Likewise, they could also show how a particular model could be used to solve different types of word problems. This will help to promote creative thinking in mathematics – different processes, one product; and one process, different products.

For average pupils, making concrete divisions of blocks and looking for relationships helps them to confidently visualise how useful and fast drawing a model is – a model often gives the answer almost instantly. For above average pupils who are familiar with using concrete parts to make comparisons, they need to be exposed to harder word problems that involve large numbers, so that concrete division of blocks becomes cumbersome and is of little use.

PSLE (Primary School Leaving Examinations) mathematics questions that use the model method are seldom as demanding as those questions set in assessment books. As a result, uninformed parents are worried that their children are not good enough to solve these challenging arithmetic word problems, which have yet to appear in the public examinations, except in mathematics contests and Olympiads. Moreover, some of these questions inappropriately use the model method, by using instead the algebraic method to arrive at the answers. Furthermore, some writers and schools have also oversold the strengths of the model method, by posing "artificially-created" questions that lend themselves nicely to the model approach.

Many of those who were part of the Dr Kho's team which first promoted the model method in Singapore, have since left the profession, or those who are still around are about to retire. Moreover, it is unfortunate that almost two decades of learning and teaching the model method to thousands of pupils, has resulted in relatively little research and teaching methodology, to help practising teachers effectively to each the mathematical models. Unless there is a follow-up team in place to continue to update and to upgrade current teachers on the effectiveness of teaching mathematical models, younger teachers will not be able to tap the experience of those who have taught the model method successfully to pupils of mixed abilities. There must be on-going in-service courses to enable mathematics educators to share with each other about their experiences - successes and difficulties - in teaching the model method to different groups of pupils.

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